Phase spaces related to standard classical *r*-matrices

1996 J. Phys. A: Math. Gen. 29 1841

(http://iopscience.iop.org/0305-4470/29/8/030)

J. Phys. A: Math. Gen. **29** (1996) 1841–1857. Printed in the UK

Phase spaces related to standard classical r-matrices

S Zakrzewski

Department of Mathematical Methods in Physics, University of Warsaw, Hoża 74, 00-682 Warsaw, Poland

Received 20 November 1995

Abstract. Fundamental representations of real simple Poisson Lie groups are Poisson actions with a suitable choice of the Poisson structure on the underlying (real) vector space. We study these (mostly quadratic) Poisson structures and corresponding phase spaces (symplectic groupoids).

0. Introduction

The recent development of non-commutative geometry and, in particular, the theory of quantum groups, raises the question of what happens with known models of physical systems when we pass from usual configurations to non-commutative ones. For classical mechanical systems, this means that we allow the configuration space to be a Poisson manifold (positions need not commute). The phase space corresponding to a usual configuration manifold (Poisson structure equal to zero) is its cotangent bundle. For a general Poisson manifold, the phase space plays the role of the corresponding symplectic groupoid (if it exists, it is unique—if one restricts to oneself connected and simply connected fibres).

It is natural first to consider mechanical systems with symmetry. In the Poisson case a symmetry is described by a Poisson action (of a Poisson group). This requirement imposes a reasonable limitation on the choice of the Poisson structure and actually leads to a construction of it.

In this paper we construct Poisson structures on real finite-dimensional vector spaces (the configuration spaces), such that the action of a chosen linear simple Poisson group becomes a Poisson action (the Poisson structure on the group is typically given by a standard classical r-matrix). We also construct the corresponding phase spaces.

1. Preliminaries and notation

For the theory of Poisson Lie groups we refer to [1–5]. Let us recall some basic notions and facts. We follow the notation used in our previous papers [6–8].

A *Poisson Lie group* is a Lie group G equipped with a Poisson structure π such that the multiplication map is Poisson. The latter property is equivalent to the following property (called *multiplicativity* of π):

$$\pi(gh) = \pi(g)h + g\pi(h) \qquad \text{for } g, h \in G.$$

Here $\pi(g)h$ denotes the right translation of $\pi(g)$ by h etc. This notation will be used throughout the paper.

0305-4470/96/081841+17$19.50

A Poisson Lie group is said to be *coboundary* if

$$\pi(g) = rg - gr \tag{1}$$

for a certain element $r \in \bigwedge^2 \mathfrak{g}$. Here $\mathfrak{g}$ denotes the Lie algebra of G. Any bi-vector field of the form (1) is multiplicative. It is Poisson if and only if

$$[r, r] \in \left(\bigwedge^3 \mathfrak{g} \right)_{\mathrm{inv}}$$

(the Schouten bracket $[r, r]$ is $\mathfrak{g}$-invariant). In this case the element r is said to be a *classical r-matrix* (*on* $\mathfrak{g}$).

If G is semisimple, any Poisson Lie group structure on G is coboundary. The standard classical r-matrix for a simple group—which corresponds to 'the standard (quantum) q-deformation'—is given by (cf [9] and proposition 2.1 in [10])

$$r = c \sum_{\alpha > 0} \frac{X_\alpha \wedge X_{-\alpha}}{\langle X_\alpha, X_{-\alpha} \rangle} \tag{2}$$

where $X_{\pm\alpha}$ are (positive and negative) root vectors relative to a Cartan subalgebra in $\mathfrak{g}$, $\langle ., . \rangle$ is the Killing form and c is a constant (if G is compact, $X_{-\alpha} = \overline{X}_\alpha$ and c is imaginary).

Let (G, π) be a Poisson Lie group. An action of G on a Poisson manifold (M, π_M) is said to be a *Poisson action* if the action map $G \times M \to M$ is Poisson. It holds if and only if the following (G, π)*-multiplicativity* of π_M is satisfied:

$$\pi_M(gx) = \pi(g)x + g\pi_M(x) \qquad \text{for } g \in G, x \in M.$$

For any fixed action $G \times M \ni (g, x) \mapsto gx \in M$ and any k-vector $w \in \bigwedge^k \mathfrak{g}$ we denote by w_M the associated k-vector field on M:

$$w_M(x) := wx.$$

2. The problem

The classical r-matrices for simple Lie groups like $SL(n, \mathbb{R})$, $SO(n, \mathbb{R})$ and $SU(n)$ are relatively well investigated (in the following we shall consider mainly the standard r-matrices (2), which indeed represent the non-trivial part of all classical r-matrices). In order to consider mechanical systems based on Poisson symmetry (typically being a 'deformation' of some ordinary symmetry), we first have to deal with the following problems.

(i) Given an action $G \times M \to M$ (the ordinary symmetry) and a Poisson structure π on G making it a Poisson Lie group (G, π) (a 'deformation' of the group), find all Poisson structures π_M on M such that the action becomes Poisson (the 'deformed' symmetry).

(ii) In cases when M plays the role of the configurational manifold, construct the phase space $\mathrm{Ph}(M, \pi_M)$ i.e. the symplectic groupoid of (M, π_M).

For symplectic groupoids, phase spaces of Poisson manifolds and so on we refer to [11–16].

For simplicity, in this paper we consider only the essential part of the structure of the symplectic groupoid (which is, in most cases, sufficient to formulate the classical model). Namely, for a given Poisson manifold (M, π_M) of dimension k we shall construct a symplectic manifold S of dimension $2k$, a surjective Poisson map from S to M and its Lagrangian section. In this case, we shall simply call S the *phase space* of (M, π_M).

3. Fundamental bi-vector field

Let $G \times M \to M$ be an action. Let $r \in \bigwedge^2 \mathfrak{g}$ be a classical r-matrix and π the corresponding Poisson structure (1) (this notation is fixed throughout the section).

Lemma 3.1. (1) r_M is (G, π)-multiplicative.

(2) Any (G, π)-multiplicative π_M is given by $\pi_M = r_M + \pi_{\mathrm{inv}}$, where π_{inv} is a G-invariant bi-vector field.

(3) $[r_M, \pi_{\mathrm{inv}}] = 0$.

(4) $[r_M, r_M] = [r, r]_M$.

Point (1) follows from $r(gx) = (rg - gr)x + g(rx)$. Point (3) follows from the fact that r_M is built out of the fundamental vector fields of the action (and these vector fields preserve π_{inv}). From (3) it follows that if both r_M and π_{inv} are Poisson then π_M is also Poisson. Point (4) follows from the known property of fundamental fields of the action:

$$[X_M, Y_M] = [X, Y]_M \qquad \text{for } X, Y \in \mathfrak{g}$$

(the Lie bracket on $\mathfrak{g}$ being defined by identifying elements of $\mathfrak{g}$ with the corresponding *right-invariant* vector fields on G).

In analogy with fundamental vector fields X_M, we call r_M the *fundamental bi-vector field*. It is essential to know whether it is Poisson.

Example 3.2. Poisson Minkowski spaces (Poincaré group action). Any invariant element of $\bigwedge^3 \mathfrak{g}$, where $\mathfrak{g} = \mathbb{R}^4 \rtimes o(1,3)$ is the Poincaré Lie algebra, is proportional to

$$\Omega = g^{jk} g^{lm} e_j \wedge e_l \wedge \Omega_{km} \qquad \Omega_{km} := e_k \otimes g(e_m) - e_m \otimes g(e_k) \in o(1,3) \tag{3}$$

(summation convention), where $(e_j)_{j=0,\ldots,3}$ is a basis in $M = \mathbb{R}^4$, g is the Lorentz metric and g^{jk} are the components of the contravariant metric (cf [8, 17]). Since

$$\Omega_M(x) = g^{jk} g^{lm} e_j \wedge e_l \wedge (e_k g(e_m, x) - e_m g(e_k, x)) = 0$$

for each classical r-matrix on $\mathfrak{g}$ the fundamental bi-vector field r_M on M is Poisson (because $[r_M, r_M] = [r, r]_M \sim \Omega_M = 0$). By point (2) of lemma 3.1 this is the only (G, π)-multiplicative bi-vector field on M, since zero is the only G-invariant bi-vector field on M. (Recall also that any Poisson structure on G comes from an r-matrix [8].) In conclusion, for each Poisson Poincaré group there is exactly one Poisson Minkowski space (see also [7]). This is also true for the case of arbitrary signature, $\mathfrak{g} = \mathbb{R}^{p+q} \rtimes o(p,q)$, in dimension $n = p + q > 3$. (Cf [18] for the quantum case.)

Example 3.3. Poisson Minkowski spaces (Lorentz group action). Classical r-matrices for the Lorentz Lie algebra $o(1,3)$ are classified in [6]. We know that $[r, r] = [r_-, r_-]$ and it is non-zero only in the case $r_- = \mathrm{i}\lambda X_+ \wedge X_-$ (in the classification of [6]) with $\lambda \neq 0$,

$$[r_-, r_-] = -\lambda^2 [X_+ \wedge X_-, X_+ \wedge X_-] = 2\lambda^2 X_+ \wedge [X_+, X_-] \wedge X_- = 4\lambda^2 X_+ \wedge H \wedge X_-$$

where X_+, X_-, H is the standard basis:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Considering the usual action of the Lorentz Lie algebra on the Minkowski space $M = \mathbb{R}^{1+3}$, we obtain

$$(X_+ \wedge X_-)_M(x) = 2\Omega_{01}(x) \wedge \Omega_{13}(x)$$

where (see (3))

$$\Omega_{jk}(x) = e_j x_k - e_k x_j.$$

Since $\Omega_{jk}(x)$, $\Omega_{kl}(x)$ and $\Omega_{lj}(x)$ are linearly dependent for each fixed j, k, l,

$$(X_+ \wedge H \wedge X_-)_M(x) = -2\Omega_{30}(x) \wedge \Omega_{01}(x) \wedge \Omega_{13}(x) = 0$$

$(H_M(x) = \Omega_{30}(x))$, but

$$(X_+ \wedge JH \wedge X_-)_M(x) = -2\Omega_{21}(x) \wedge \Omega_{01}(x) \wedge \Omega_{13}(x)$$

(J is the complex structure in $\mathfrak{g}$) is not zero. It follows that r_M is Poisson if and only if λ^2 is real, i.e. either α or β in [6] has to be zero. Moreover, since the only Lorentz invariant bi-vector field on M is zero, r_M is the only (G, π)-multiplicative field on M. It follows that for $\alpha \cdot \beta \neq 0$ there is no Poisson structure on M such that the action is Poisson. (A similar fact should hold for quantum Lorentz groups [19]: q should be real or of modulus one.)

Returning to a general technique, now consider two special cases of r-matrices.

3.1. The triangular case: $[r, r] = 0$

Let $\xi: T^*M \to M$ be the cotangent bundle projection and let π_0 denote the canonical Poisson structure of T^*M. In the triangular case:

(i) r_M is Poisson (by lemma 3.1(4));

(ii) r_{T^*M} is Poisson (also lemma 3.1(4)); $\xi_* r_{T^*M} = r_M$;

(iii) $\pi_{T^*M} := r_{T^*M} + \pi_0$ is Poisson (by lemma 3.1(3)); $\xi_* \pi_{T^*M} = r_M$.

This means that problems formulated in section 2 are relatively easily solved. For the phase space one can take the open subset of points in T^*M, in which the Poisson structure π_{T*M} is non-degenerate (it is certainly non-degenerate in a neighbourhood of the zero section—that is why we have added π_0 in (iii). (To construct the symplectic groupoid one should still find the foliation symplectically orthogonal to the fibres of the projection and choose points which also have the projection on M along this foliation.)

For another approach to this case, see [16].

3.2. The case of a simple $\mathfrak{g}$

In this case one can use the method of [20] to rewrite the condition $[r_M, r_M] = 0$. Denote by Ω the canonical invariant element of $\bigwedge^3 \mathfrak{g}$. Its Killing transported version to $\bigwedge^3 \mathfrak{g}^*$ is defined by

$$\Omega^\dagger(X, Y, Z) = \langle [X, Y], Z \rangle.$$

It is known [21] that all invariant elements of $\bigwedge^3 \mathfrak{g}$ are proportional to Ω, hence $[r, r] \sim \Omega$. Suppose $[r, r]$ is not zero. Then r_M is Poisson $\Leftrightarrow$ $\Omega_M = 0$ (in general, Ω_M is just G-invariant). Now, $\Omega x = 0 \Leftrightarrow$ the composition of linear maps

$$\mathbb{R} \xrightarrow{\Omega} \bigwedge^3 \mathfrak{g} \to \bigwedge^3 (\mathfrak{g}/\mathfrak{g}_x)$$

is zero $\Leftrightarrow$ the composition of linear maps

$$\mathbb{R} \xleftarrow{\Omega^*} \bigwedge^3 \mathfrak{g} \leftarrow \bigwedge^3 (\mathfrak{g}_x)^0$$

is zero $\Leftrightarrow$ the composition of linear maps

$$\mathbb{R} \xleftarrow{\Omega^\dagger} \bigwedge^3 \mathfrak{g} \leftarrow \bigwedge^3 (\mathfrak{g}_x)^\perp$$

is zero $\Leftrightarrow \Omega^\dagger|_{(\mathfrak{g}_x)^\perp} = 0 \Leftrightarrow \langle [X, Y], Z\rangle = 0$ for $X, Y, Z \in \mathfrak{g}_x^\perp \Leftrightarrow [\mathfrak{g}_x^\perp, \mathfrak{g}_x^\perp] \subset \mathfrak{g}_x$. Concluding,

$$[r_M, r_M]x = 0 \Longleftrightarrow [\mathfrak{g}_x^\perp, \mathfrak{g}_x^\perp] \subset \mathfrak{g}_x. \tag{4}$$

The advantage of this method is that we do not have to use the explicit form of Ω.

Proposition 3.4. In the following three cases, for any classical r-matrix on $\mathfrak{g}$ the fundamental bi-vector field r_M on M is Poisson:

(1) $\mathfrak{g} = so(n, \mathbb{R})$, $M = \mathbb{R}^n$
(2) $\mathfrak{g} = sl(n, \mathbb{R})$, $M = \mathbb{R}^n$
(3) $\mathfrak{g} = sp(n, \mathbb{R})$, $M = \mathbb{R}^{2n}$.

For $\mathfrak{g} = su(n)$, $M = \mathbb{C}^n = \mathbb{R}^{2n}$, the fundamental bi-vector field is Poisson if and only if r is triangular.

Proof. Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. The dual basis in $(\mathbb{R}^n)^*$ is denoted by $e^1, \ldots, e^n$. We consider subsequent cases separately.

(1) $\mathfrak{g} = so(n, \mathbb{R})$. Set $x = e_n$, then $\mathfrak{g}_x$ is spanned by Ω_{jk} for $j, k < n$ and $\mathfrak{g}_x^\perp$ is spanned by Ω_{jn} for $j < n$. It is easy to see that $[\Omega_{jn}, \Omega_{kn}]$ is proportional to Ω_{jk}, hence it belongs to $\mathfrak{g}_x$. This shows (by equation (4)) that $[r_M, r_M] = 0$.

(2) $\mathfrak{g} = sl(n, \mathbb{R})$. For the same $x = e_n$, $\mathfrak{g}_x$ is spanned by ${e_n}^j$ (the matrix units) for $j < n$, but $[{e_n}^j, {e_n}^k] = 0$.

(3) $\mathfrak{g} = sp(n, \mathbb{R})$. We use the basis $e_1, \ldots, e_n, e^1, \ldots, e^n$ in $M = \mathbb{R}^n \oplus (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$. We have four types of matrix units defined by

$$e_j{}^k := e_j \otimes e^k \qquad e^j{}_k := e^j \otimes e_k \qquad e_{jk} := e_j \otimes e_k \qquad e^{jk} := e^j \otimes e^k$$

with the action on $(x, p) \in M = \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ given explicitly by

$$e_j{}^k(x, p) = e_j x^k \qquad e^j{}_k(x, p) = e^j p_k \qquad e_{jk}(x, p) = e_j p_k \qquad e^{jk}(x, p) = e^j x^k.$$

We use the following basis in $\mathfrak{g} = sp(n, \mathbb{R})$:

$$a_{jk} := e_{jk} + e_{kj} \ (j \leqslant k) \qquad b^{jk} := e^{jk} + e^{kj} \ (j \leqslant k) \qquad d_j{}^k := e_j{}^k - e^k{}_j. \tag{5}$$

For $x := e_n$, $\mathfrak{g}_x$ is spanned by a_{jk}, b^{jk} with $j, k < n$ and $d_j{}^k$ with $k < n$ and $\mathfrak{g}_x^\perp$ is spanned by a_{jn}, $d_n{}^j$ with $j = 1, \ldots n$. Now, $[a_{jn}, a_{kn}] = 0$ and $[d_n{}^j, a_{kn}] = \delta_k^j a_{nn} + \delta_n^j a_{kn} \in \mathfrak{g}_x$.

We now pass to the case of $\mathfrak{g} = su(n)$, with the basis

$$F_j{}^k := e_j{}^k - e_k{}^j \qquad G_j{}^k := \mathrm{i}(e_j{}^k + e_k{}^j) \qquad H_j := e_{j+1}{}^{j+1} - e_j{}^j. \tag{6}$$

For $x = e_n$, $\mathfrak{g}_x$ is spanned by $F_j{}^k$, $G_j{}^k$ for $j, k < n$ and H_j for $j < n - 1$ and $\mathfrak{g}_x^\perp$ contains $F_j{}^n$, $G_j{}^n$ for $j < n$ (and the component of H_{n-1}, orthogonal to H_j for $j < n - 1$). We have

$$[F_1{}^n, G_1{}^n] = 2\mathrm{i}(e_1{}^1 - e_n{}^n) \notin \mathfrak{g}_x.$$

Actually one can see that

$$[\mathfrak{g}_x^\perp, \mathfrak{g}_x^\perp] = \mathfrak{g}_x + \langle \mathrm{i}(e_1{}^1 - e_n{}^n)\rangle$$

(inclusion (4) is violated only by one dimension). $\square$

4. Standard r-matrices for $sl(n,\mathbb{R})$ and $sp(n,\mathbb{R})$

According to equation (2), the standard r-matrix for $\mathfrak{g} = sl(n,\mathbb{R})$ is given by

$$r = \varepsilon \sum_{j<k} e_j{}^k \wedge e_k{}^j \qquad (\varepsilon \in \mathbb{R}) \tag{7}$$

(the Cartan subalgebra consists of diagonal matrices and the 'positive' roots are contained in the upper-triangular matrices). Considering the natural action of $\mathfrak{g}$ on $M := \mathbb{R}^n$, we obtain

$$r_M(x) = \varepsilon \sum_{j<k} x^j x^k e_j \wedge e_k = \sum_{j<k} (x^j e_j) \wedge (x^k e_k) \tag{8}$$

which defines the following Poisson brackets of coordinates

$$\{x^j, x^k\} = \varepsilon x^j x^k \qquad \text{for } j < k. \tag{9}$$

For $\tilde{\mathfrak{g}} := sp(n,\mathbb{R})$ we choose $d_j{}^k$ $(j < k)$ and a_{jk} $(j \leqslant k)$ as positive roots, which gives the following expression for the standard r-matrix (we denote it by $\tilde{r}$):

$$\tilde{r} = \varepsilon \left(\sum_{j<k} d_j{}^k \wedge d_k{}^j + \tfrac{1}{2} \sum_{j,k} a_{jk} \wedge b^{jk} \right)$$

(notation as in (5)). Considering the natural action of $\tilde{\mathfrak{g}}$ on $\tilde{M} = T^*M = \mathbb{R}^n \oplus (\mathbb{R}^n)^*$, we obtain

$$\begin{aligned} \tilde{r}_{\tilde{M}}(x,p) &= \varepsilon \left[\sum_{j<k} (e_j x^k - e^k p_j) \wedge (e_k x^j - e^j p_k) + \tfrac{1}{2} \sum_{j,k} (e_j p_k + e_k p_j) \wedge (e^j x^k + e^k x^j) \right] \\ &= \varepsilon \left[x \wedge p + \sum_{j<k} (x^j x^k e_j \wedge e_k - p_j p_k e^j \wedge e^k) \right. \\ &\quad \left. + \sum_j \left(\sum_k (1 - \operatorname{sgn}(k-j)) x^k p_k \right) e_j \wedge e^j \right] \end{aligned} \tag{10}$$

which gives the following quadratic Poisson brackets of coordinates and momenta:

$$\{x^j, x^k\} = \varepsilon x^j x^k \qquad \{p_j, p_k\} = -\varepsilon p_j p_k \qquad \text{for } j < k \tag{11}$$

$$\{x^j, p_k\} = \varepsilon \left[x^j p_k + \delta^j{}_k \left(\sum_i (1 - \operatorname{sgn}(i-j)) x^i p_i \right) \right]. \tag{12}$$

Now observe that we can add to (10) the canonical bi-vector π_0 on T^*M which modifies (12) in the following way:

$$\{x^j, p_k\} = -\delta^j{}_k + \varepsilon \left[x^j p_k + \delta^j{}_k \left(\sum_i (1 - \operatorname{sgn}(i-j)) x^i p_i \right) \right]. \tag{13}$$

The Poisson structure (11) and (13) projects on (9) and is non-degenerate in a neighbourhood of $(\mathbb{R}^n \oplus \{0\}) \cup (\{0\} \oplus (\mathbb{R}^n)^*)$. We have thus constructed the phase space for (M, r_M) (M is indeed embedded into this phase space as a Lagrangian submanifold: $p_j = 0$ are *first class* constraints).

The quantum version of the above construction has been described in [22].

Remark 4.1. The natural embedding of $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$ (the lift to T^*M), given by $e_j{}^k \mapsto d_j{}^k$, is a homomorphism of Lie bi-algebras. It follows that the action of $SL(n,\mathbb{R})$ on T^*M is Poisson.

Remark 4.2. Set $\nabla_j := x^j e_j$ (no summation). According to (8), $r_M = \sum_{j<k} \nabla_j \wedge \nabla_k$. We see that r_M is built of commuting vector fields and, actually, $r_M = \rho_M$, where ρ is a r-matrix on the abelian Lie algebra spanned by those fields. Since ρ is triangular and even abelian [23], one can easily construct the phase space of (M, ρ_M) using the method described in section 3.1, or, another method, described in [23]. In the present paper we shall exploit only the original r-matrix, because such an approach can be generalized and applied to more difficult situations (see the following sections).

5. Crossed product phase spaces and quasi-triangularity

Let $\mathfrak{g}$ be a Lie subalgebra of End V, where $V = \mathbb{R}^n$. Any classical r-matrix on $\mathfrak{g}$ may be identified as an element of $\bigwedge^2$ End V, which can be expressed in terms of matrix units:

$$r = \sum_{jklm} r^{jk}_{lm} e_j{}^l \otimes e_k{}^m \qquad r^{jk}_{lm} = -r^{kj}_{ml}.$$

If r is triangular, the phase space of (V, r_V) can be realized on $T^*V = V \times V^*$ with the Poisson structure π_{T^*V} being the sum of the canonical Poisson structure π_0 and r_{T^*V} (cf section 3.1). We then have

$$\pi_{T^*V}(x, p) = \pi_0 + \sum_{jklm} r^{jk}_{lm}(e_j x^l - p_j e^l) \otimes (e_k x^m - p_k e^m)$$

which leads to the following Poisson brackets:

$$\{x^j, x^k\} = \sum_{lm} r^{jk}_{lm} x^l x^m \qquad \{p_l, p_m\} = \sum_{jk} p_j p_k r^{jk}_{lm} \tag{14}$$

$$\{x^k, p_l\} = -\delta^k{}_l + \sum_{jm} p_j r^{jk}_{lm} x^m. \tag{15}$$

We shall also use the following abbreviated notation:

$$\{x_1, x_2\} = r x_1 x_2 \qquad \{p_1, p_2\} = p_1 p_2 r \qquad \{x_1, p_2\} = -I + p_1 r x_2. \tag{16}$$

Definition 5.1. Let (M, π_M) and (N, π_N) be two Poisson manifolds and $P := M \times N$. If π_P is a Poisson structure on P such that the cartesian projections $P \to M, N$ are Poisson, then (P, π_P) is said to be a *crossed product* of (M, π_M) and (N, π_N).

We see that in the triangular case, the phase space is a crossed product of (V, r_V) and (V^*, r_{V^*}). Note that the Poisson brackets between x^k and p_l (the cross-relations) are expressed in terms of the same r-matrix as r_V and r_{V^*}.

In the case of the non-triangular r-matrix (7) for $sl(n, \mathbb{R})$, the phase space turns out to be also a crossed product of (V, r_V) and (V^*, r_{V^*}). The Poisson structure (11) and (13) is realized on T^*V and has the following form,

$$\pi_{T^*V} = \pi_0 + r_{T^*V} + \Delta \tag{17}$$

where Δ is some additional quadratic term (in the cross-relations), which was not present in the triangular case.

We shall now explain the nature of the additional term Δ in a general situation. We assume that our r-matrix is such that r_V and r_{V^*} are Poisson (with the explicit form of Poisson brackets given by (14)) and we ask what conditions should satisfy a bi-vector field Δ of the form

$$\Delta(x, p) = \sum_{jklm} p_j \Delta^{jk}_{lm} x^m e_k \wedge e^l \tag{18}$$

on T^*V, in order to make (17) a Poisson bi-vector field. Since $r_{T^*V}(x, p) = r_V(x) + r_{V^*}(p) + \sum_{jklm} p_j r^{jk}_{lm} x^m e_k \wedge e^l$, we have

$$\pi_{T^*V}(x, p) = \pi_0 + r_V + r_{V^*} + \sum_{jklm} p_j w^{jk}_{lm} x^m e_k \wedge e^l$$

where $w^{jk}_{lm} = r^{jk}_{lm} + \Delta^{jk}_{lm}$. We look therefore for conditions on w^{jk}_{lm} under which the brackets

$$\{x_1, x_2\} = r x_1 x_2 \qquad \{p_1, p_2\} = p_1 p_2 r \qquad \{p_1, x_2\} = I - p_1 w x_2 \tag{19}$$

(abbreviated notation) satisfy the Jacobi identity. We first consider only the quadratic part

$$\{x_1, x_2\} = r x_1 x_2 \qquad \{p_1, p_2\} = p_1 p_2 r \qquad \{p_1, x_2\} = -p_1 w x_2 \tag{20}$$

(it is easy to show that the quadratic part itself must also define a Poisson bracket). Since

$$\{p_1, \{x_2, x_3\}\} = \{p_1, r_{23} x_2 x_3\} = -p_1 r_{23}(w_{12} + w_{13}) x_1 x_3$$

and

$$\begin{aligned}\{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\} &= \{-p_1 w_{12} x_2, x_3\} - \{p_1 w_{13} x_3, x_2\} \\ &= p_1(w_{13} w_{12} - w_{12} r_{23}) x_2 x_3 - (p_1(w_{12} w_{13} - w_{13} r_{32}) x_2 x_3)\end{aligned}$$

the part of the Jacobi identity corresponding to the equality $\{p_1, \{x_2, x_3\}\} = \{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\}$ is equivalent to

$$p_1([w_{12}, w_{13}] + [w_{12}, r_{23}] + [w_{13}, r_{23}]) x_2 x_3 = 0. \tag{21}$$

Similarly, the part of the Jacobi identity corresponding to $\{\{p_1, p_2\}, x_3\} = \{p_1, \{p_2, x_3\}\} - \{p_2, \{p_1, x_3\}\}$ is equivalent to

$$p_1 p_2([r_{12}, w_{13}] + [r_{12}, w_{23}] + [w_{13}, w_{23}]) x_3 = 0. \tag{22}$$

Theorem 5.2. Let $(\mathfrak{g}, r)$ be quasi-triangular, i.e. there exists an invariant symmetric element s of $\mathfrak{g} \otimes \mathfrak{g}$ such that $w := r + s$ satisfies the classical Yang–Baxter equation:

$$[[w, w]] := [w_{12}, w_{13}] + [w_{12}, w_{23}] + [w_{13}, w_{23}] = 0.$$

Then the brackets (20) satisfy the Jacobi identity (we assume that (14) already satisfy the Jacobi identity).

Proof. Since s is invariant,

$$[w_{12}, s_{23}] + [w_{13}, s_{23}] = 0$$

(for any w), hence $[[w, w]] = 0$ if and only if

$$[w_{12}, w_{13}] + [w_{12}, r_{23}] + [w_{13}, r_{23}] = 0. \tag{23}$$

This obviously implies (21). Similarly, since $[s_{12}, w_{13}] + [s_{12}, w_{23}] = 0$, $[[w, w]]$ is zero if and only if

$$[r_{12}, w_{13}] + [r_{12}, w_{23}] + [w_{13}, w_{23}] = 0$$

which implies (22). □

Remark 5.3. Let r, s and w satisfy the assumptions of theorem 5.2. For any $\lambda \in \mathbb{R}$, the element

$$w_\lambda := r + s + \lambda I \otimes I \in \operatorname{End} V \otimes \operatorname{End} V \tag{24}$$

satisfies the Yang–Baxter equation and the proof of theorem 5.2 works also for w_λ (once w satisfies (23), w_λ also satisfies (23)). This means that one can replace w by w_λ in (20).

Theorem 5.4. Under assumptions of theorem 5.2, brackets (19) satisfy the Jacobi identity if and only if

$$s^{jk}_{lm} = s^{kj}_{lm}. \tag{25}$$

Brackets (19) with w replaced by w_λ satisfy the Jacobi identity if and only if

$$(s_\lambda)^{jk}_{lm} = (s_\lambda)^{kj}_{lm} \tag{26}$$

where $s_\lambda = s + \lambda I \otimes I$.

Proof. In the part of the Jacobi identity corresponding to $\{p_1, \{x_2, x_3\}\} = \{\{p_1, x_2\}, x_3\} - \{\{p_1, x_3\}, x_2\}$, we must take care of the linear terms (the cubic terms were taken into account in the previous theorem). This gives

$$rx - (rx)^{\mathrm{t}} = wx - (wx)^{\mathrm{t}} \tag{27}$$

where $(rx)^{jk}_l := \sum_m r^{jk}_{lm} x^m$, $((rx)^{\mathrm{t}})^{jk}_l := (rx)^{kj}_l$, etc. Of course, (27) means that $sx = (sx)^{\mathrm{t}}$, i.e. (25). The modification $w \mapsto w + \lambda I \otimes I$, $s \mapsto s + \lambda I \otimes I$ leading to condition (26) is straightforward. The remaining part of the Jacobi identity leads to the same condition. □

Example 5.5. It is convenient to consider (7) as a r-matrix on $\mathfrak{g} = gl(n, \mathbb{R})$, because it is easy to write down a natural invariant symmetric element (trace form) of $\mathfrak{g} \otimes \mathfrak{g}$. Taking this element with the appropriate coefficient,

$$s = \varepsilon \sum_{j,k} e_j{}^k \otimes e_k{}^j$$

we obtain

$$w = r + s = \varepsilon \left(\sum_j e_j{}^j \otimes e_j{}^j + 2 \sum_{j<k} e_j{}^k \otimes e_k{}^j \right)$$

which satisfies the classical Yang–Baxter equation. There is a unique modification $s_\lambda = s + \lambda I \otimes I$ of s satisfying the symmetry (26), namely for $\lambda = \varepsilon$:

$$s_\lambda = s_\varepsilon = s + \varepsilon I \otimes I \qquad (s_\varepsilon)^{jk}_{lm} = \varepsilon(\delta^j_m \delta^k_l + \delta^j_l \delta^k_m).$$

Poisson brackets (19) with $w_\lambda = w_\varepsilon$ coincide with (11) and (13).

In conclusion, if $(\mathfrak{g}, r)$ is quasi-triangular and if there exists a modification $s_\lambda = s + \lambda I \otimes I$ of s satisfying (26), then one can realize the phase space of (V, r_V) on T^*V with the Poisson structure (17), where Δ is given by (18) with

$$\Delta^{jk}_{lm} = (s_\lambda)^{jk}_{lm}. \tag{28}$$

It is convenient to introduce the following notation. For each

$$\rho = \sum_{jklm} \rho^{jk}_{lm} e_j{}^l \otimes e_k{}^m \in \operatorname{End} V \otimes \operatorname{End} V \tag{29}$$

we denote by ρ_{VV^*} the bi-vector field on $T^*V = V \oplus V^*$ defined by

$$\rho_{VV^*}(x, p) = \sum_{jklm} p_j \rho^{jk}_{lm} x^m e_k \wedge e^l. \tag{30}$$

Using this notation we can write (17) as follows:

$$\pi_{T^*V} = \pi_0 + r_{T^*V} + (s_\lambda)_{VV^*} = \pi_0 + r_V + r_{V^*} + (w_\lambda)_{VV^*}. \tag{31}$$

6. $SO(n,\mathbb{R})$, imaginary quasi-triangularity and the reality condition

In this section we construct the phase space of (V, r_V), where $V := \mathbb{R}^n$, for a standard r-matrix on $so(n,\mathbb{R})$, using methods which are completely analogous to those used in [24] for the investigation of a real differential calculus on quantum Euclidean spaces.

In terms of the 'angular momentum' generators $M_j{}^k := e_j{}^k - e_k{}^j$, the standard r-matrix on $\mathfrak{g} = so(n,\mathbb{R})$ is given by

$$r = \frac{\varepsilon}{4}\sum_{j<k}(M_j{}^k + M_{j'}{}^{k'}) \wedge (M_j{}^{k'} + M_k{}^{j'}) = \varepsilon \sum_{j<k<j'} M_j{}^k \wedge M_k{}^{j'} \tag{32}$$

where $j' := n+1-j$ (the underlying Cartan subalgebra consists of anti-diagonal matrices in this case).

The above r-matrix is not quasi-triangular in the real sense (this is a characteristic feature of compact simple groups). Instead, one can find an invariant symmetric element s of $\mathfrak{g}\otimes\mathfrak{g}$ such that $w := r \pm \mathrm{i}s$ satisfies the classical Yang–Baxter equation. We say that $(\mathfrak{g}, r)$ is *imaginary quasi-triangular* in this case. In our case,

$$s = \frac{\varepsilon}{2}\sum_{j,k} M_j{}^k \otimes M_k{}^j = \varepsilon\sum_{j,k}(e_j{}^k \otimes e_k{}^j - e_j{}^k \otimes e_j{}^k)$$

(the simplest way to obtain w is to extract the first-order term from the known R-matrix for the quantum $SO(n)$ group). In order to satisfy (26), we add $\lambda I \otimes I$ to s with $\lambda = \varepsilon$:

$$s_\lambda = \varepsilon\sum_{j,k}(e_j{}^k \otimes e_k{}^j - e_j{}^k \otimes e_j{}^k + e_j{}^j \otimes e_k{}^k).$$

It is clear that (19) with w replaced by $w_\lambda := r - \mathrm{i}s_\lambda$ defines a complex-valued Poisson structure on $T^*V \cong \mathbb{R}^{2n}$. Equally well we can treat (19) as defining a holomorphic Poisson structure on the complexification $(T^*V)^{\mathbb{C}} \cong \mathbb{C}^{2n}$ of T^*V. In the following we shall construct a real form of this holomorphic Poisson manifold, playing the role of the phase space of (V, r_V).

We first derive Poisson brackets of coordinates with basic $\mathfrak{g}$-invariant functions,

$$x^2 := g_{jk}x^jx^k \qquad p^2 := p_jp_kg^{jk} \qquad E := \langle p, x\rangle = p_jx^j$$

(summation convention assumed), where g_{jk} is the metric tensor (equal to the Kronecker delta in our orthonormal basis e_j). Since s is universal (the Killing form), independent of r (only the coefficient at s depends on the proportionality constant between $[r, r]$ and the canonical element of $\bigwedge^3 \mathfrak{g}$), the part

$$\{p_j, x^k\}_{\mathrm{univ}} = \delta_j{}^k + \mathrm{i}\varepsilon(E\delta_j{}^k + p_jx^k - x_jp^k) \tag{33}$$

of Poisson brackets (19) corresponding to $\pi_0 - \mathrm{i}(s_\lambda)_{VV^*}$ is universal. It follows that the brackets with $\mathfrak{g}$-invariant functions are also universal (these functions are Casimirs of r_{T^*V} and it is sufficient to use only $\pi_0 - \mathrm{i}(s_\lambda)_{VV^*}$). From (33), it is now easy to obtain the following brackets:

$$\begin{aligned}
&\{x^2, x^j\} = 0 \qquad \{\tfrac{1}{2}p^2, x^j\} = p^j + \mathrm{i}\varepsilon p^2x^j\\
&\{p^2, p^j\} = 0 \qquad \{p_j, \tfrac{1}{2}x^2\} = x_j + \mathrm{i}\varepsilon x^2p_j\\
&\{E, x^j\} = x^j + 2\mathrm{i}\varepsilon Ex^j - \mathrm{i}\varepsilon x^2p^j \qquad \{p_j, E\} = p_j + 2\mathrm{i}\varepsilon Ep_j - \mathrm{i}\varepsilon p^2x_j.
\end{aligned} \tag{34}$$

Let us now introduce one more invariant function:

$$\Lambda := 1 + 2\mathrm{i}\varepsilon E - \varepsilon^2x^2p^2 \tag{35}$$

(following the method of [24]). From (34) it follows that

$$\{\Lambda, x^j\} = 2\mathrm{i}\varepsilon\Lambda x^j \qquad \{p_j, \Lambda\} = 2\mathrm{i}\varepsilon p_j \Lambda. \tag{36}$$

Denote by $\mathrm{Hol}(Y)$ the algebra of holomorphic functions on a complex manifold Y. Recall that any holomorphic map $\phi: Y \to Z$ (of complex manifolds) defines a linear multiplicative map $\Phi: \mathrm{Hol}(Z) \to \mathrm{Hol}(Y)$ by the pullback: $\Phi(f) = f \circ \phi$. Similarly, any anti-holomorphic map $\psi: Y \to Z$ defines an anti-linear multiplicative map $\Psi: \mathrm{Hol}(Z) \to \mathrm{Hol}(Y)$ by pullback followed by the complex conjugation: $\Psi(f) = \overline{f \circ \phi}$. Using Λ, we define an anti-holomorphic map ψ from

$$P^{\mathbb{C}} := \{(x, p) \in (T^*V)^{\mathbb{C}} : \Lambda \neq 0\} \subset (T^*V)^{\mathbb{C}}$$

into $(T^*V)^{\mathbb{C}}$ by

$$\Psi(x^j) = x^j \qquad \Psi(p_j) = \frac{p_j + \mathrm{i}\varepsilon p^2 x_j}{\Lambda}. \tag{37}$$

Since

$$\Psi(p^2) = \frac{p^2}{\Lambda} \qquad \Psi(E) = \frac{E + \mathrm{i}\varepsilon p^2 x^2}{\Lambda} \qquad \Psi(\Lambda) = \frac{1}{\Lambda} \tag{38}$$

the underlying map ψ maps $P^{\mathbb{C}}$ into $P^{\mathbb{C}}$. Moreover, since $\Psi(\Psi(x^j)) = x^j$ and

$$\Psi(\Psi(p_j)) = \Psi\left(\frac{p_j + \mathrm{i}\varepsilon p^2 x_j}{\Lambda}\right) = \Lambda\left(\frac{p_j + \mathrm{i}\varepsilon p^2 x_j}{\Lambda} - \mathrm{i}\varepsilon \frac{p^2}{\Lambda} x_j\right) = p_j$$

the anti-holomorphic map $\psi: P^{\mathbb{C}} \to P^{\mathbb{C}}$ is an involution. Therefore we can define the corresponding *real form* P of $P^{\mathbb{C}}$ as the set of fixed points of ψ:

$$P := \{z \in P^{\mathbb{C}} : \psi(z) = z\}. \tag{39}$$

The antilinear multiplicative involution Ψ corresponding to the map ψ will be henceforth denoted by a star:

$$(x^j)^* = x^j \qquad (p_j)^* = \frac{p_j + \mathrm{i}\varepsilon p^2 x_j}{\Lambda}. \tag{40}$$

Let us collect once more the basic formulae (38):

$$(p^2)^* = \frac{p^2}{\Lambda} \qquad E^* = \frac{E + \mathrm{i}\varepsilon p^2 x^2}{\Lambda} \qquad \Lambda^* = \frac{1}{\Lambda}$$

$$P := \{(x, p) : (x^j)^* = \overline{x^j}, (p_j)^* = \overline{p_j}\}.$$

The fundamental theorem of this section says that the star operation is compatible with the Poisson brackets (19).

Theorem 6.1. The Poisson structure (19) is real with respect to the star operation (40):

$$\{f, g\}^* = \{f^*, g^*\}. \tag{41}$$

Proof. We have to prove (41) in two cases: (i) $f = p_j$, $g = x^k$ and (ii) $f = p_j$, $g = p_k$. The case $f = x^j$, $g = x^k$ is trivial.

(i) We set $T_j := p_j + \mathrm{i}\varepsilon p^2 x_j$. Since

$$\{p_j, x^k\}^* = \delta_j{}^k - (p_l)^* (\overline{w_\lambda})^{lk}_{jm} x^m$$

and

$$\{(p_j)^*, x^k\} = \left\{\frac{T_j}{\Lambda}, x^k\right\} = \frac{1}{\Lambda}(\{T_j, x^k\} - 2\mathrm{i}\varepsilon T_j x^k)$$

we have to show that

$$\{T_j, x^k\} - 2\mathrm{i}\varepsilon T_j x^k = \Lambda \delta_j{}^k - T_l (\overline{w_\lambda})^{lk}_{jm} x^m. \tag{42}$$

Since

$$\{T_j, x^k\} = \delta_j{}^k - p_l (w_\lambda)^{lk}_{jm} x^m + \mathrm{i}\varepsilon(2x_j p^k + 2\mathrm{i}\varepsilon p^2 x_j x^k + p^2 g_{jl} r^{lk}_{mn} x^m x^n)$$

the left-hand side of (42) equals

$$\delta_j{}^k - p_l (w_\lambda)^{lk}_{jm} x^m + \mathrm{i}\varepsilon(2(x_j p^k - p_j x^k) + p^2 g_{jl} r^{lk}_{mn} x^m x^n)$$

whereas the right-hand side of (42) equals

$$(1 + 2\mathrm{i}\varepsilon E - \varepsilon^2 x^2 p^2)\delta_j{}^k - (p_l + \mathrm{i}\varepsilon p^2 x_l)(\overline{w_\lambda})^{lk}_{jm} x^m.$$

Since

$$p_l((w_\lambda)^{lk}_{jm} - (\overline{w_\lambda})^{lk}_{jm})x^m = 2p_l(-\mathrm{i}s_\lambda)^{lk}_{jm} x^m = -2\mathrm{i}\varepsilon(E\delta_j{}^k + p_j x^k - x_j p^k)$$

(cf (33)), it follows that (42) is equivalent to

$$\mathrm{i}\varepsilon p^2 g_{jl} r^{lk}_{mn} x^m x^n = -\varepsilon^2 x^2 p^2 \delta_j{}^k - \mathrm{i}\varepsilon p^2 x_l (\overline{w_\lambda})^{lk}_{jm} x^m$$

or

$$\mathrm{i} g_{jl} r^{lk}_{mn} x^m x^n = -\varepsilon x^2 \delta_j{}^k - \mathrm{i} x_l (\overline{w_\lambda})^{lk}_{jm} x^m.$$

Taking into account

$$g_{jl} r^{lk}_{mn} = -g_{lm} r^{lk}_{jn} \tag{43}$$

(r^{lk}_{mn} belongs to $\mathfrak{g}$ with respect to indices l, m), it means that (42) is equivalent to

$$\mathrm{i} g_{lm} (\mathrm{i}s_\lambda)^{lk}_{jn} x^m x^n = \varepsilon x^2 \delta_j{}^k$$

which can be easily verified.

(ii) Since

$$\{T_j, \Lambda\} = 2\mathrm{i}\varepsilon T_j \Lambda$$

we have

$$\left\{\frac{T_j}{\Lambda}, \frac{T_k}{\Lambda}\right\} = \frac{\{T_j, T_k\}}{\Lambda^2}.$$

Therefore it is sufficient to show that

$$\{T_j, T_k\} = T_l T_m r^{lm}_{jk}.$$

We have

$$\begin{aligned}\{T_j, T_k\} = \{p_j, p_k\} &+ \mathrm{i}\varepsilon p^2(\{p_j, x_k\} - \{p_k, x_j\}) + (\mathrm{i}\varepsilon)^2\{p^2 x_j, p^2 x_k\} \\ &= p_l p_m r^{lm}_{jk} + \mathrm{i}\varepsilon p^2[-p_l r^{la}_{jm} x^m g_{ak} + p_l r^{la}_{km} x^m g_{aj} + 2\mathrm{i}\varepsilon(p_j x_k - x_j p_k)] \\ &\quad -\varepsilon^2 p^2[p^2\{x_j, x_k\} + 2(p_k x_j - x_k p_j)].\end{aligned}$$

Since

$$-p_l r^{la}_{jm} x^m g_{ak} = p_l x_m r^{lm}_{jk}$$

(by the argument similar to (43)), we have finally

$$\begin{aligned}\{T_j, T_k\} &= p_l p_m r^{lm}_{jk} + \mathrm{i}\varepsilon p^2 p_l x_m (r^{lm}_{jk} - r^{lm}_{kj}) + (\mathrm{i}\varepsilon p^2)^2 x_l x_m r^{lm}_{jk} \\ &= (p_l + \mathrm{i}\varepsilon p^2 x_l)(p_m + \mathrm{i}\varepsilon p^2 x_m) r^{lm}_{jk}.\end{aligned}$$

□

Corollary. P is endowed with a structure of a real analytic Poisson manifold. If f_0, g_0 are two real analytic functions on P, then their Poisson bracket is defined by

$$\{f_0, g_0\} := \{f, g\}|_P$$

where f, g are the (local) holomorphic extensions of f_0, g_0 to $P^{\mathbb{C}}$ (by (41), the restriction of $\{f, g\}$ to P is real).

P is the required phase space of (V, r_V).

7. Poisson action of $SU(n)$ on $\mathbb{C}^n$

Here we treat $V = \mathbb{C}^n$ as a real manifold ($V \cong \mathbb{R}^{2n}$). Specifying (2) to the case of $SU(n)$ we get the following standard r-matrix (see (6) for the basis of $su(n)$)

$$r = \varepsilon \tfrac{1}{2} \sum_{j<k} (e_j{}^k - e_k{}^j) \wedge J(e_j{}^k + e_k{}^j) \tag{44}$$

where $J: V \to V$ is the complex structure of V (multiplication by the imaginary unit). From now on we set $\varepsilon = 1$ (arbitrary ε will be restored in the final formulae). It is convenient to work with the complexification $V^{\mathbb{C}} \cong V \oplus \mathrm{i}V$ and the complex-linear embedding

$$V \ni z \mapsto z^{\mathbb{C}} := \tfrac{1}{2}(z - \mathrm{i}Jz) \in V^{\mathbb{C}}.$$

We have

$$z = z^{\mathbb{C}} + \overline{z^{\mathbb{C}}} \qquad Jz = \mathrm{i}(z^{\mathbb{C}} - \overline{z^{\mathbb{C}}})$$

and the typical notation

$$(e_k)^{\mathbb{C}} = \left(\frac{\partial}{\partial x_k}\right)^{\mathbb{C}} = \frac{\partial}{\partial z_k} = \partial_k.$$

Note that

$$e_j{}^k z = (e_j{}^k z)^{\mathbb{C}} + \overline{(e_j{}^k z)^{\mathbb{C}}} = (e_j z^k)^{\mathbb{C}} + \overline{(e_j z^k)^{\mathbb{C}}} = z^k \partial_j + \bar{z}^k \bar{\partial}_j.$$

In this notation, the fundamental bi-vector field r_V is as follows,

$$r_V(z) = \mathrm{i} \sum_{jk} \mathrm{sgn}(k-j)(\tfrac{1}{2}\nabla_j \wedge \nabla_k - \tfrac{1}{2}\bar{\nabla}_j \wedge \bar{\nabla}_k + |z^j|^2 \partial_k \wedge \bar{\partial}_k) \tag{45}$$

where $\nabla_k := z^k \partial_k$, $\bar{\nabla}_k := z^k \bar{\partial}_k$.

Lemma 7.1. $[r_V, r_V](z) = -\|z\|^2 Jz \wedge \pi_0$, where

$$\pi_0 = 2\mathrm{i} \sum_k \partial_k \wedge \bar{\partial}_k$$

is the canonical constant bi-vector on $V = \mathbb{C}^n = \mathbb{R}^{2n}$.

Proof. Taking into account

$$\begin{aligned}
[|z^j|^2 \partial_k \wedge \bar{\partial}_k, \nabla_a \wedge \nabla_b] &= \bar{z}^j(-\bar{\partial}_k) \wedge [z^j \nabla_k, \nabla_a \wedge \nabla_b] \\
&= -\bar{z}^j \bar{\partial}_k \wedge [(z^j \delta_k^a \partial_a - z^a \delta_a^j \partial_k) \wedge \nabla_b + \nabla_a \wedge (z^j \delta_k^b \partial_b - z^b \delta_b^j \partial_k)] \\
&= -|z^j|^2 \bar{\partial}_k \wedge \partial_k \wedge [(\delta_k^a - \delta_a^j)\nabla_b - (\delta_k^b - \delta_b^j)\nabla_a]
\end{aligned}$$

and

$$\begin{aligned}
[|z^j|^2 \partial_k \wedge \bar{\partial}_k, |z^a|^2 \partial_b \wedge \bar{\partial}_b] &= [z^j \partial_k \wedge \bar{z}^j \bar{\partial}_k, z^a \partial_b \wedge \bar{z}^a \bar{\partial}_b] \\
&= [z^j \partial_k, z^a \partial_b] \wedge \bar{z}^j \bar{\partial}_k \wedge \bar{z}^a \bar{\partial}_b + z^j \partial_k \wedge z^a \partial_b \wedge [\bar{z}^j \bar{\partial}_k, \bar{z}^a \bar{\partial}_b] \\
&= z^j \partial_k \wedge z^a \partial_b \wedge (\bar{z}^j \delta_k^a \bar{\partial}_b - \bar{z}^a \delta_b^j \bar{\partial}_k) + \mathrm{CC}
\end{aligned}$$

(+CC means 'plus complex conjugated terms'), we see that $[r_V, r_V](z)$ equals

$$
\begin{aligned}
-\sum_{jkab} & \operatorname{sgn}(k-j)\operatorname{sgn}(b-a) \\
& \times\{|z^j|^2[(\delta^a_k-\delta^j_a)\nabla_b-(\delta^b_k-\delta^j_b)\nabla_a]+2|z^a|^2\delta^j_b\nabla_b\}\wedge\partial_k\wedge\bar\partial_k+\mathrm{CC} \\
= & -2\sum_{jkab}\operatorname{sgn}(k-j)\operatorname{sgn}(b-a) \\
& \times[|z^j|^2(\delta^a_k-\delta^j_a)\nabla_b+|z^a|^2\delta^j_b\nabla_b]\wedge\partial_k\wedge\bar\partial_k+\mathrm{CC} \\
= & -2\sum_{kb}\sum_{ja}\operatorname{sgn}(k-j)\operatorname{sgn}(b-a) \\
& \times[|z^j|^2(\delta^a_k-\delta^j_a)+|z^a|^2\delta^j_b]\nabla_b\wedge\partial_k\wedge\bar\partial_k+\mathrm{CC}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{ja} & \operatorname{sgn}(k-j)\operatorname{sgn}(b-a)(|z^j|^2\delta^a_k-|z^j|^2\delta^j_a+|z^a|^2\delta^j_b)\nabla_b\wedge\partial_k\wedge\bar\partial_k \\
&= \sum_j |z^j|^2(\operatorname{sgn}(k-j)\operatorname{sgn}(b-k)+\operatorname{sgn}(b-k)\operatorname{sgn}(j-b) \\
&\quad +\operatorname{sgn}(j-b)\operatorname{sgn}(k-j))
\end{aligned}
$$

and

$$\operatorname{sgn}(k-j)\operatorname{sgn}(b-k)+\operatorname{sgn}(b-k)\operatorname{sgn}(j-b)+\operatorname{sgn}(j-b)\operatorname{sgn}(k-j)=-1$$

for $b\neq k$. It follows that

$$[r_V,r_V](z)=2\|z\|^2\sum_b(\nabla_b-\bar\nabla_b)\wedge\sum_k\partial_k\wedge\bar\partial_k=-\|z\|^2\sum_b \mathrm{i}(\nabla_b-\bar\nabla_b)\wedge\sum_k 2\mathrm{i}\partial_k\wedge\bar\partial_k.$$

□

Corollary. For any classical r-matrix $\tilde r$ on $\mathfrak{g}=su(n)$ there is a constant c such that $[\tilde r_V,\tilde r_V]=-c\|z\|^2Jz\wedge\pi_0$. The Poisson structures π_V on V for which the action of $SU(n)$ on V is Poisson are exactly bi-vector fields

$$\pi_V=\tilde r_V+\Delta \tag{46}$$

such that the bi-vector field Δ on V is $SU(n)$-invariant and satisfies

$$[\Delta,\Delta](z)=c\|z\|^2Jz\wedge\pi_0. \tag{47}$$

It is easy to show that all $SU(n)$-invariant bi-vector fields Δ on V are of the following form:

$$\Delta=\tfrac{1}{2}a\pi_0+\tfrac{1}{2}bz\wedge Jz \tag{48}$$

where $a=a(\|z\|^2)$ and $b=b(\|z\|^2)$ are arbitrary functions of $\|z\|^2$. We shall write condition (47) in terms of these functions.

Lemma 7.2. $[\Delta,\Delta](z)=\|z\|^2Jz\wedge\pi_0$ if and only if

$$aa'+b(a-a't)=t. \tag{49}$$

Here $t\equiv\|z\|^2$ and prime means differentiating with respect to the variable t.

Proof. If K, L are bi-vector fields and f, g are functions, then

$$[fK, gL] = fg[K, L] - fK(\mathrm{d}g) \wedge -gK \wedge L(\mathrm{d}g) \tag{50}$$

where by $K(\mathrm{d}g)$ we denote the contraction of K with $\mathrm{d}g$ on the first place. In particular,

$$[fK, fK] = f^2 - 2fK \wedge K(\mathrm{d}f).$$

Using

$$\pi_0(\tfrac{1}{2}\mathrm{d}\|z\|^2) = -Jz \qquad (z \wedge Jz)(\tfrac{1}{2}\mathrm{d}\|z\|^2) = \|z\|^2 Jz$$

and

$$[\pi_0, z \wedge Jz] = 2Jz \wedge \pi_0$$

we obtain

$$[\tfrac{1}{2}a\pi_0, \tfrac{1}{2}a\pi_0] = aa' Jz \wedge \pi_0$$
$$[bz \wedge Jz, bz \wedge Jz] = 0$$
$$[\tfrac{1}{2}a\pi_0, \tfrac{1}{2}bz \wedge Jz] = \tfrac{1}{2}b(a - a'\|z\|^2)Jz \wedge \pi_0.$$

From this, (49) follows immediately. □

Of course, the easy way to solve (49) is to write

$$b = \frac{t - aa'}{a - a't} \tag{51}$$

but in this way we have no control of regularity over these functions and we do not see the simplest cases. To pick up the simplest cases, let us consider Δ at most quadratic, i.e. $a = a_0 + a_1 t$, $b = b_0$, where a_0, a_1, b_0 are some constants. Inserting this form of a and b in (49) gives the following two cases:

(i) $a_0 = 0$, $a_1 = \pm 1$, b_0 arbitrary;

(ii) a_0 arbitrary, $a_1 = \pm 1$, $b_0 = \mp 1$.

One of the simplest non-quadratic solutions for Δ is the following solution of degree four:

(iii) $a = h = \text{constant} \neq 0$, $b = t/h$.

An example of a non-singular rational solution is given by $a = 1 - t^2$ (in this case the denominator of (51) is positive: $a - a't = 1 + t^2$).

Another way to pick up a simple case is to assume that Δ is (as r_V) tangential to the spheres $\|z\| = \text{constant}$. It is easy to show that this conditions holds if and only if $a = bt$. In this case (49) reduces to

$$ab = t.$$

It means that $a = \pm t$, $b = \pm 1$. This is a special case of type (i) above.

We end by listing the explicit form of the Poisson brackets corresponding to the mentioned cases. From the general form (46) and (48) with the standard r-matrix (44), we obtain

$$\{z^j, z^k\} = \mathrm{i}\varepsilon z^j z^k \qquad \text{for } j < k \tag{52}$$

$$\{z^j, \bar{z}^k\} = -\mathrm{i}\varepsilon b z^j \bar{z}^k \qquad \text{for } j \neq k \tag{53}$$

$$\{z^j, \bar{z}^j\} = \mathrm{i}\varepsilon \sum_k \mathrm{sgn}(j - k) \cdot |z^k| + \mathrm{i}\varepsilon a - \mathrm{i}\varepsilon b |z^j| \tag{54}$$

(we have restored the parameter ε). Now we list the cases which seem to be most interesting.

(1) *Poisson $SU(n)$-spheres*. According to the discussion above, there are only two Poisson structures on $\mathbb{C}^n$ solving our problem and tangential to the spheres, namely

$$\{z^j, z^k\} = \mathrm{i}\varepsilon z^j z^k \qquad \text{for } j < k$$
$$\{z^j, \bar{z}^k\} = -\mathrm{i}\sigma\varepsilon b z^j \bar{z}^k \qquad \text{for } j \neq k$$
$$\{z^j, \bar{z}^j\} = \mathrm{i}\varepsilon(\sigma\|z\|^2 - \sigma|z|^j + \sum_k \mathrm{sgn}(j-k)\cdot|z^k|) = 2\sigma\mathrm{i}\varepsilon \sum_{\sigma k<\sigma j} |z^k|^2$$

where $\sigma = \pm 1$. The function $z \mapsto \|z\|^2$ is a Casimir function of this Poisson structure (and can be fixed, which leads to a sphere S^{2n-1}).

(2) *Twisted annihilation and creation 'operators'*. Setting $h = \varepsilon a_0$ in case (ii) above, we obtain

$$\{z^j, z^k\} = \mathrm{i}\varepsilon z^j z^k \qquad \text{for } j < k$$
$$\{z^j, \bar{z}^k\} = \mathrm{i}\sigma\varepsilon b z^j \bar{z}^k \qquad \text{for } j \neq k$$
$$\{z^j, \bar{z}^j\} = \mathrm{i}h + \mathrm{i}\varepsilon(\sigma\|z\|^2 + \sigma|z|^j + \sum_k \mathrm{sgn}(j-k)\cdot|z^k|) = \mathrm{i}h + 2\sigma\mathrm{i}\varepsilon \sum_{\sigma k\leqslant\sigma j} |z^k|^2$$

where $\sigma = \pm 1$. This is the Poisson version of the 'twisted canonical commutation relations' of [25] (see also [26]). It may describe the phase space of a Poisson deformed harmonic oscillator.

(3) The non-quadratic brackets corresponding to case (iii) above are given by

$$\{z^j, z^k\} = \mathrm{i}\varepsilon z^j z^k \qquad \text{for } j < k \tag{55}$$
$$\{z^j, \bar{z}^k\} = -\mathrm{i}\frac{\varepsilon}{h}\|z\|^2 z^j \bar{z}^k \qquad \text{for } j \neq k \tag{56}$$
$$\{z^j, \bar{z}^j\} = \mathrm{i}\varepsilon h + \mathrm{i}\varepsilon\left(-\frac{1}{h}\|z\|^2|z^j|^2 + \sum_k \mathrm{sgn}(j-k)\cdot|z^k|\right). \tag{57}$$

Problem. What is the quantum counterpart of condition (49)? What is the quantum counterpart of relations (56) and (57)?

References

[1] Drinfeld V G 1983 Hamiltonian structures on Lie groups, Lie bi-algebras and the meaning of the classical Yang–Baxter equations *Sov. Math. Dokl.* **27** 68–71

[2] Drinfeld V G 1986 Quantum groups *Proc. ICM, Berkeley* vol 1, pp 789–820

[3] Semenov-Tian-Shansky M A 1985 Dressing transformations and Poisson Lie group actions *Publ. Res. Inst. Math. Sci., Kyoto University* **21** 1237–60

[4] Lu J-H and Weinstein A 1990 Poisson Lie groups, dressing transformations and Bruhat decompositions *J. Diff. Geom.* **31** 501–26

[5] Lu J-H 1990 Multiplicative and affine Poisson structures on Lie groups *PhD Thesis* University of California, Berkeley

[6] Zakrzewski S 1994 Poisson structures on the Lorentz group *Lett. Math. Phys.* **32** 11–23

[7] Zakrzewski S 1995 Poisson homogeneous spaces *'Quantum Groups, Formalism and Applications', Proc. XXX Winter School on Theoretical Physics (14–26 February 1994, Karpacz)* ed J Lukierski, Z Popowicz and J Sobczyk (Warsaw: Polish Scientific Publishers PWN) pp 629–39

[8] Zakrzewski S 1995 Poisson Poincaré groups *'Quantum Groups, Formalism and Applications', Proc. XXX Winter School on Theoretical Physics (14–26 February 1994, Karpacz)* ed J Lukierski, Z Popowicz and J Sobczyk (Warsaw: Polish Scientific Publishers PWN) pp 433–9

[9] Belavin A and Drinfeld V G 1984 Triangle equations and simple Lie algebras *Sov. Sci. Rev. Math.* **4** 93–165

[10] Jimbo M 1986 Quantum *R* matrix related to generalized Toda system: an algebraic approach *Lect. Notes Phys.* **246** 335–61

[11] Coste A, Dazord P and Weinstein A 1987 Groupoïdes symplectiques *Publications du Départment de Mathématiques* Université Claude Bernard Lyon I

[12] Karasev M V 1987 Analogues of objects of Lie group theory for nonlinear Poisson brackets *Math. USSR-Izv.* **28** 497–527

[13] Zakrzewski S 1990 Quantum and classical pseudogroups. Part I and II *Comm. Math. Phys.* **134** 347–95

[14] Weinstein A 1991 Noncommutative geometry and geometric quantization *Actes du colloque en l'honneur de Jean-Marie Souriau (Aix-en-Provence 1990) (Progr. Math. 99)* (Boston, MA: Birkhäuser) pp 446–61

[15] Zakrzewski S 1993 Symplectic models of groups with non-commutative spaces *Proc. 12th Winter School on Geometry and Topology (11–18 January 1992, Srni) (Supplemento ai rendiconti del Circolo Matematico di Palermo, serie II, No 32)* pp 185–94

[16] Xu P 1993 Poisson manifolds associated with group actions and classical triangular r-matrices *J. Funct. Anal.* **112** 218–40

[17] Zakrzewski S 1996 Poisson structures on the Poincaré group *Preprint* Warsaw, January, q-alg/9602001.

[18] Podleś P and Woronowicz S L On the classification of quantum Poincaré groups hep-th/9412059 *UC Berkeley Preprint* PAM-632

[19] Woronowicz S L and Zakrzewski S 1994 Quantum deformations of the Lorentz group. The Hopf *-algebra level *Compositio Mathematica* **90** 211–43

[20] Gurevich D and Panyushev D 1994 On Poisson pairs associated to modified R-matrices *Duke Math. J.* **73** 249–55

[21] Greub W, Halperin S and Van Stone R 1976 *Curvature, Connection and Cohomology (Pure and Applied Mathematics 47)* (New York: Academic) theorem I, p 189

[22] Zumino B 1991 Deformation of the quantum mechanical phase space with bosonic or fermionic coordinates *Berkeley Preprint* LBL-30120, UCB-PTH-91/1

[23] Zakrzewski S 1994 Geometric quantization of Poisson groups—diagonal and soft deformations *Proc. Taniguchi Symp. Symplectic Geometry and Quantization Problems (Sanda, 1993) (Contemporary Mathematics 179)* ed Y Maeda, H Omori and A Weinstein pp 271–85

[24] Ogievetsky O and Zumino B 1992 Reality in the differential calculus on q-Euclidean spaces *Lett. Math. Phys.* **25** 121–30

[25] Pusz W and Woronowicz S L 1989 Twisted second quantization *Rev. Mod. Phys.* **27** 231

[26] Shabanov S q-oscillators, (non-)Kähler manifolds and constrained systems *Preprint* hep-th/9403042